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Upper Bound on the Decay of Correlations in the Plane Rotator Model with Long-Range Random Interaction

P. Picco[1]

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We give an upper bound on the decay of correlation function for the plane rotator model with Hamiltonian

$$-\frac{1}{2}\sum_{xy}\frac{J_{xy}\cos(\theta_x-\theta_y)}{|x-y|^{(3/2+\varepsilon)d}}$$

in dimension $d=1$ and $d=2$ when (J_{xy}) are independent random variables with mean zero.

KEY WORDS: Random interaction; random variables; long range; spin glass.

1. INTRODUCTION

This paper is a continuation of an earlier work where we proved the absence of breakdown of symmetry for classical *xy* spin glass model in two dimensions with long-range interaction in a region where the corresponding ferromagnetic model has a spontaneous magnetization. Here we give an upper bound on the decay of the two-point correlation function. In the nonrandom case, when there is no spontaneous magnetization, upper bounds were already given by Fisher and Jasnow,[(2)] McBryan and Spencer,[(3)] Schlossman,[(4)] Bonato, Klein, and Perez,[(5)] Ito,[(6)] and Messager, Miracle-Sole, and Ruiz.[(7)]

Let us remark that in Refs. 3 and 6 it was assumed that the Hamiltonian is reflection positive [8] in order to compare the decay of the two-point correlation function with the large distance behavior of a lattice Green function.

[1] Centre de Physique Théorique, C.N.R.S.—Luminy—Case 907, 13288 Marseille Cedex 9, France.

In the case of a spin glass model the Hamiltonian does not have this property. In Ref. 7 Messager *et al.* do not use the reflection positivity and obtain the best results in all cases where the spontaneous magnetization is known to be zero, e.g., Pfister.[9] Their theorems are always true in the spin glass case as long as the decay of the potential is strong enough and the coupling constant J_{xy} are bounded random variables. However, their theorem cannot be applied in the case considered in Ref. 1. Moreover, we are able to treat the case where the coupling constants J_{xy} are unbounded random variables with some mild restrictions on the moments of order $l \geqslant 3$. Similar restrictions was imposed by Khanin and Sinai.[10] This is a nontrivial improvement of Ref. 1 where only the case of *bounded* sub-Gaussian random variables was considered. We also treat the one-dimensional case, since Khanin[11] asserted without giving a proof that there is no phase transition almost surely. We give an upper bound on the decay of the two-point correlation function for this model.

2. DESCRIPTION OF THE MODELS, MAIN RESULTS, AND STRATEGY OF THE PROOFS

We consider the classical xy spin glass model in one and two dimensions.

Let Λ be a finite subset of $\mathbb{Z}^d$ we define

$$H_\Lambda(\theta_\Lambda, \theta_{\Lambda^c}) = -\sum_{x\in\Lambda}\sum_{y\in\mathbb{Z}^d\setminus\{x\}} \frac{J_{xy}\cos(\theta_x-\theta_y)}{|x-y|^{(3/2+\varepsilon)d}} \tag{2.1}$$

where in (2.1) each spin θ_x takes its values in the torus. We assume that $(J_{xy})_{(x,y)\in\mathbb{Z}^{2d}}$ are independent identically distributed random variables satisfying the following conditions:

$$\mathbb{E}(J_{xy}) = 0$$

$$\mathbb{E}(J_{xy}^2) = \sigma^2$$

$$\mathbb{E}(J_{xy}^p) \leqslant p!\,\frac{\sigma^2}{2}H^{p-2} \qquad \text{for some fixed } H\in\mathbb{R}^+$$

Let us denote by $\langle\ \rangle(J)$ the expectation with respect to *any* Gibbs state corresponding to the Hamiltonian 2.1.

The main results of this paper are the following theorems.

Theorem 2.1. If $d=1$ and if β is large enough then there exists a constant $K_\varepsilon(\beta)>0$ such that

$$\operatorname{Prob}\left(\overline{\lim_{L\to\infty}}\,\frac{\log|\langle\cos(\theta_0-\theta_L)\rangle|}{\log L} \leqslant -K_\varepsilon(\beta)\right) = 1 \tag{2.2}$$

Theorem 2.2. If $d=2$ and if β is large enough then there exists a constant $K'_\varepsilon(\beta)>0$ such that

$$\text{Prob}\left[\overline{\lim_{L\to\infty}}\ \frac{\log|\langle\cos(\theta_0-\theta_l)\rangle|}{(\log L)^\gamma}\leqslant -K'_\varepsilon(\beta)\right]=1 \tag{2.3}$$

where $\gamma=1-\varepsilon_1$ for some ε_1 arbitrary small but nonzero.

Remarks. (1) Theorem 2.2 can be formulated as follows: Let Ω be the set of random bounds $(J_{xy})_{(x,y)\in\mathbb{Z}^4}$ and μ the corresponding measure on Ω. If β is large enough, then one can find a constant K'_ε and a subset Ω_0 of Ω with $\mu(\Omega_0)=0$ in such a way that the following proprty is true: for any $\varepsilon_2>0$, any $J\in\Omega\backslash\Omega_0$ there exists a constant $L_0=L_0(J,\varepsilon_2)$ such that

$$|\langle\cos(\theta_0-\theta_L|\rangle(J)|\leqslant \exp-(\mathbf{K}'_\varepsilon(\beta)+\varepsilon_2)(\log L)^\gamma \tag{2.4}$$

for all

$$L\geqslant L_0(J,\varepsilon_2).$$

(2) In the case $\varepsilon=1/2$, Theorem 2.2 gives $\exp-(K/\beta)(\log L)^{1-\varepsilon_1}$ for arbitrary small ε_1, as upper bound, which is better than the upper bound in the ferromagnetic case which is derived in Ref. 7, namely, $\exp-(K/\beta)\log\log L$.

The proof is based on the McBryan–Spencer technics[3] using an idea of Messager *et al.*[7] for the choice of complex translation. Let us first recall the following proposition of McBryan and Spencer.[3]

Proposition 2.3. Let $\Lambda(L)$ be a square box centered at the origin of side $2L+1$ and let L denote also the point of the x_1 axis with $x_1=L$. For a given family of real numbers $\{a_x\}$ indexed by $x\in\Lambda(L)$ the following inequality holds:

$$|\langle\cos(\theta_0-\theta_L)\rangle(\beta,J)|\leqslant\exp\Bigg\{-(a_L-a_0)$$
$$+\beta\max_\theta\left[\left|\sum_{\substack{x,y\\x\in\Lambda(L)}}\frac{J_{xy}}{|x-y|^{(3/2+\varepsilon)d}}\cos(\theta_x-\theta_y)[\cosh(a_x-a_y)-1]\right|\right]\Bigg\}$$

We refer the reader to Ref. 3 for the proof of this proposition. See also Ref. 6, p. 752.

The good choice for a_x is

$$a_x = a_{|x|} = K(\beta) \sum_{r=|x|}^{L} \frac{1}{r(\log^+ r)^{1-\gamma}} \qquad \text{if } |x| \leqslant L$$
$$a_x = 0 \qquad \text{if } |x| > L \tag{2.6}$$

where $|x| = \max(|x_2|, \ |x_d|)$ and $\log^+ r = \max(1, \log r)$. $K(\beta)$ and γ are constants which will be chosen later. It is easy to see that $a_L - a_0 \sim [K(\beta)/1-\gamma](\log L)^\gamma$ if $\gamma > 0$. Therefore it is sufficient to choose γ [resp. $K(\beta)$] in such a way that

$$f_L(J, \beta, \gamma) = \frac{\beta}{(\log L)^\gamma} \max_\theta \left| \sum_{\substack{x,y \\ x \in \Lambda_L}} \frac{J_{xy} \cos(\theta_x - \theta_y)}{|x-y|^{(3/2+\varepsilon)d}} \{\cosh(a_x - a_y) - 1\} \right| \tag{2.7}$$

is bounded, almost surely as $L \to \infty$, by a constant $f(\gamma, \beta)$ [resp. $K(\beta) - f(\gamma, \beta) \geqslant K_\varepsilon(\beta)$ for β large enough] in order to prove Theorem 2.2.

Let us denote

$$\Delta H \mid \theta(\Lambda_2)\, \theta(\Lambda_2)$$
$$= \sum_{\substack{x \in \Lambda_1 \\ y \in \Lambda_2 \\ x \neq y}} \frac{J_{xy}}{|x-y|^{(3/2+\varepsilon)d}} \cos(\theta_x - \theta_y)[\cosh(a_x - a_y) - 1] \tag{2.8}$$

then

$$f_L(J, \gamma, \beta) = \frac{\beta}{(\log L)^\gamma} \max_\theta |2\Delta H(\theta(\Lambda(L)), \theta(\Lambda(L))) + \Delta H(\theta(\Lambda(L)), \theta(\Lambda^c(L)))| \tag{2.9}$$

Since the proofs of Theorems 2.1 and 2.2 are long we decompose them in four steps. Let us explain the strategy which is based on summation by blocks as in Ref. 1 but with more computational complexity.

Step 1. We consider in 2.9 the term $\Delta H(\theta(\Lambda(L)), \theta(\Lambda^c(L^2)))$. If the random variables J_{xy} are bounded by $\tilde{J}$ we can bound $\Delta H(\theta(\Lambda(L), \theta(\Lambda^c(L^2)))$ uniformly with respect to J and θ by $\tilde{J} L^{-2\varepsilon d}$ using only the decay of the potential. If the random varaibles are unbounded we use the following inequality:

$$|\Delta H(\theta(\Lambda(L)), \theta(\Lambda^c(L^2)))| \leqslant \sum_{\substack{x \in \Lambda(L) \\ y \in \Lambda^c(L^2)}} \frac{|J_{xy}|}{|x-y|^{3+2\varepsilon}} \{\cosh(a_x) - 1\} \tag{2.10}$$

Therefore

$$|\Delta H(\theta(\Lambda), \theta(\Lambda^c))| \leqslant \sum_{\substack{x \in \Lambda \\ y \in \Lambda^c(L^2)}} \frac{(|J_{xy}| - \mathbb{E}(|J_{xy}|))}{|x-y|^{(3+2\varepsilon)}} \{\cosh(a_x) - 1\} + \sum_{\substack{x \in \Lambda(L) \\ y \in \Lambda^c(L^2)}} \frac{\mathbb{E}(|J_{xy}|)\{\cosh(a_x) - 1\}}{|x-y|^{3+2\varepsilon}} \tag{2.11}$$

as we will see in Step 2 it is sufficient to consider the case $K(\beta) < 1$.

Then we prove the following lemma:

Lemma 2.4. For any $K(\beta) < 1$,

(i) $$\lim_{L \to \infty} \sum_{\substack{x \in \Lambda(L) \\ y \in \Lambda^c(L^2)}} \frac{\mathbb{E}(|J_{xy}|)}{|x-y|^{3+2\varepsilon}} \{\cosh(a_x) - 1\} \tag{2.12}$$

(ii) $$\lim_{L \to \infty} \sum_{\substack{x \in \Lambda(L) \\ y \in \Lambda^c(L^2)}} \frac{(|J_{xy}| - \mathbb{E}(|J_{xy}|))}{|x-y|^{(3+2\varepsilon)}} \{\cosh(a_x) - 1\} = 0 \quad \text{almost surely} \tag{2.13}$$

Remark. It is crucial that the set Ω_0 of measure zero where 2.13 is not true does not depend on the spin configuration θ. If Ω_0 would depend on θ the nondenumerable reunion $\bigcup_\theta \Omega_0(\theta)$ could be of measure 1.

Step 2. We consider the term $\Delta H(\theta(\Lambda(L)), \theta(\Lambda(L^2) \backslash \Lambda(2L)))$. This is the first intermediate region. If we use an inequality similar to (2.11), this sum,

$$\sum_{\substack{x \in \Lambda(L) \\ y \in \Lambda(L^2) \backslash \Lambda(2L)}} \frac{\mathbb{E}(|J_{xy}|)}{|x-y|^{3+2\varepsilon}} \{\cosh a_x - 1\}$$

goes to infinity with L. To solve this problem we discretize $\cos(\theta_x - \theta_y)$ (not the θ_x as in Ref. 1) in the following way: Since $\cos(\theta_x - \theta_y) = \cos(\theta_x)\cos(\theta_y) + \sin\theta_x \sin\theta_y$ and $|\cos\theta_x| \in [0\ \ 1]$, we can perform a dyadic expansion of $\cos\theta_x$:

$$\cos\theta_x = \sigma_0(\theta_x) \left(\sum_{K=1}^{\infty} \frac{\tau_K(\theta_x)}{2^K} \right) \tag{2.14}$$

where $\tau_K(\theta_x) = 0$ or 1 and $\sigma_0(\theta_x) = \text{sign}(\cos\theta_x)$. Defining $\tilde{\sigma}_K(\theta_x) = 2\tau_K(\theta_x) - 1$ if $K \geqslant 1$ $\tilde{\sigma}_0(\theta_x) = 1$, we get

$$\cos\theta_x = \sigma_0(\theta_x) \left[\frac{1}{2} + \sum_{K=1}^{\infty} \frac{\tilde{\sigma}_K(\theta_x)}{2^{K+1}} \right] \tag{2.15}$$

If we set $\sigma_K(\theta_x) = \sigma_0(\theta_x)\,\tilde{\sigma}_K(\theta_x)$ we get

$$\cos\theta_x = \sum_{K=0}^{\infty} \frac{\sigma_K(\theta_x)}{2^{K+1}} \tag{2.16}$$

For an integer M, we define $\cos^{(M)}\theta_x = \sum_{K=0}^{M} [\sigma_K(\theta_x)/2^{K+1}]$. We consider in $\Delta H(\theta(\Lambda(L)),\ \theta(\Lambda(L^2)\backslash\Lambda(2L)))$ only the terms corresponding to $\cos\theta_x \cos\theta_y$ since the terms with the sine are treated in the same way for simplicity we call these terms $\Delta H(\theta(\Lambda(L)),\ \theta(\Lambda(L^2)\backslash\Lambda(2L)))$ and let $\Delta H^{(M)}(\theta(\Lambda(L),\ \theta(\Lambda(L^2)\backslash\Lambda(2L)))$ be the same terms with $\cos\theta_x$ replaced by $\cos^M\theta$.

It is straightforward that

$$|\Delta H - \Delta H^{(M)}| \leqslant \frac{1}{2^M} \sum_{\substack{x\in\Lambda(L)\\ y\in\Lambda(L^2)\backslash\Lambda(2L)}} \frac{|J_{xy}|}{|x-y|^{(3+2\varepsilon)}} [\cosh a_x - 1] \tag{2.17}$$

Denoting $\Delta H(\Lambda(L), \Lambda(L^2)\backslash\Lambda(2L), |J|)$ the sum in the right-hand side of 2.17) we get

$$\begin{aligned} &|\Delta H(\theta(\Lambda(L)), \theta(\Lambda(L^2)\backslash\Lambda(2L)))| \\ &\quad\leqslant \frac{1}{2^M} \Delta H(\Lambda(L), \Lambda(L^2)\backslash\Lambda(2L), |J' - \mathbb{E}(|J|)) \\ &\quad\quad + \frac{1}{2^M} \Delta H(\Lambda(L), \Lambda(L^2)\backslash\Lambda(2L), \mathbb{E}(|J|)) \\ &\quad\quad + |\Delta H^{(M)}(\theta(\Lambda(L), \theta(\Lambda(L^2)\backslash\Lambda(2L))| \end{aligned} \tag{2.18}$$

Lemma 2.5. There exists a choice of M as function of L and d such that

(i) $$\lim_{L\to\infty} \frac{1}{2^M} \Delta H(\Lambda(L), \Lambda(L^2)\backslash\Lambda(2L), \mathbb{E}(|J|)) = 0 \tag{2.19}$$

(ii) $$\lim_{L\to\infty} \frac{1}{2^M} \Delta H(\Lambda(L)\, \Lambda(L^2)\backslash\Lambda(2L), |J| - \mathbb{E}(|J|)) = 0 \tag{2.20}$$

almost surely

(iii) if $K(\beta) < d_{(1/2-\delta)}$ with $0 < \delta < \varepsilon$

$$\text{Prob}\left(\overline{\lim_{L\to\infty}}\ \max_{\theta} |\Delta H^{(M)}(\theta(\Lambda(L)), \theta(\Lambda(L^2)\backslash\Lambda(2L))) = 0\right) = 1 \tag{2.21}$$

Therefore $\Delta H(\theta(\Lambda(L),\ \theta(\Lambda(L^2)\backslash\Lambda(2L)))$ goes to zero as L goes to infinity uniformly with respect to θ and almost surely with respect to J.

Step 3. We consider the term $\Delta H(\theta(\Lambda(L), \theta(\Lambda(2L)))$. Let us remark that if we used the inequality (2.11) then the term

$$\sum_{\substack{x \in \Lambda(L) \\ y \in \Lambda(2L)\backslash\{x\}}} \frac{\mathbb{E}(|J_{xy}|)}{|x-y|^{(3+2\varepsilon)}}, \{\cosh(a_x - a_y) - 1\}$$

goes to infinity with L. Therefore we use a discretization as in Step 2.

First we subdivide $\Lambda(2L)\backslash\Lambda(L)$ and $\Lambda(L)$ into concentric crowns as follows:

For any integer L, there exists a unique integer $m(L)$ such that $2^{m(L)} \leqslant L < 2^{m(L)+1}$. We write m instead of $m(L)$ when there is no danger of confusion. If X is a real number let $[X]$ be its integer part. For any integer $K \geqslant 1$ and any $j \in 1, 2,..., [L2^{-m+K}] - 1 \equiv \alpha_K(L) - 1$ let

$$\mathscr{C}_j^{K+1} = \{x \in \mathbb{Z}^2 / 2^{m-K}(j-1) < |x| \leqslant 2^{m-K} j\}$$

$$\mathscr{C}_{\alpha_K(L)}^{K+1} = \Lambda_L \Big\backslash \bigcup_{j=1}^{\alpha_K(L)-1} \mathscr{C}_j^{K+1}$$

If $j \in \alpha_K(L)+1,..., 2\alpha_K(L)-1$, let

$$\mathscr{C}_j^{K+1} = \{x \in \mathbb{Z}^2 / L + (j-1+\alpha_K(L))2^{m-K} < L + (j - \alpha_K(L))2^{m-K}\}$$

and

$$\mathscr{C}_{2\alpha_K(L)}^{K+1} = \Lambda(2L) \Big\backslash \Big\{ \Lambda(L) \bigcup_{j=\alpha_K(L)+1}^{2\alpha_K(L)-1} \mathscr{C}_j^{K+1} \Big\}$$

Let us remark that the width of $\mathscr{C}_{\alpha_K(L)}^{K+1} j = \alpha_K(L)+1$ and $\mathscr{C}_{2\alpha_K(L)}^{K+1}$ is less than 2^{m-K+1} and no less than 2^{m-K}. The reason of such decomposition will become clear in the sequel. Roughly speaking for a given m and K, except for the two crowns $\mathscr{C}_{\alpha_K(L)}^{K+1}$ and $\mathscr{C}_{2\alpha_K(L)}^{K+1}$, the width of the crowns $\mathscr{C}_j^{K+1}$ is constant (and equal to 2^{m-K}) if L increases from 2^m to $2^{m+1}-1$. In Ref. 1 we allow the width of these crowns to increase with L. Here the number of crowns $\mathscr{C}_j^{K+1}$ increase with L. According to this decomposition $\Delta H(\theta(\Lambda(L)),$ $\theta(\Lambda(2L))$ can be written

$$\Delta H(\theta(\Lambda(L)), \theta(\Lambda(2L)))$$
$$= \sum_{j=1}^{\alpha_K(L)} \Delta H(O(\mathscr{C}_j^{K+1}), \theta(\mathscr{C}_j^{K+1})) + 2\Delta H(\theta(\mathscr{C}_j^{K+1}), \theta(\mathscr{C}_{j+1}^{K+1})) + \sum_{p=2}^{K+1} R_p \tag{2.22}$$

where for a given $p > 2$, R_p denotes

$$R_p = \sum_{j=1}^{\alpha_{p-1}(L)} 2\Delta H(\theta(\mathscr{C}_j^p), \theta(\mathscr{C}_{j+2}^p)) + \sum_{j=1}^{\alpha_{p-2}(L)} 2\Delta H(\theta(\mathscr{C}_{2j-1}^p), \theta(\mathscr{C}_{2j+2}^p))$$
$$+ 2\Delta H(\theta(\mathscr{C}_{2\alpha_{p-2}(L)}^p), \theta(\mathscr{C}_{2\alpha_{p-2}+3}^p)) \tag{2.23}$$

where the last term in (2.23) occurs only if α_{p-1} is odd:

$$R_2 = \sum_{j=1}^{\alpha_1(L)} \sum_{K=j+2}^{\alpha_2(L)} 2\Delta H(\theta(\mathscr{C}_j^2), \theta(\mathscr{C}_K^2))$$

In this step we consider only $\sum_{p=2}^{K} R_p$. We first discretize the cosine as in Step 2. With self-explaining notations we get

$$\sum_{p=2}^{K} R_p(J, \cos\theta)$$
$$= \sum_{p=2}^{K} R_p(J, \cos^M \theta) + \sum_{p=2}^{K} [R_p(J, \cos\theta) - R_p(J, \cos^M \theta)] \tag{2.24}$$

It is straightforward that the modulus of the last sum in (2.24) does not exceed

$$\frac{1}{2^M} \sum_{p=2}^{K} R_p(|J|) = \frac{1}{2^M} \sum_{p=2}^{K} R_p(|J| - \mathbb{E}(|J|)) + \frac{1}{2^M} \sum_{p=2}^{K} R_p(\mathbb{E}(|J|)) \tag{2.25}$$

where $R_p(|J|)$ is nothing byt $R_p(J, \cos\theta)$ with J_{xy} replaced by $|J_{xy}|$ and $\cos\theta$ replaced by 1.

We prove the following lemma:

Lemma 2.6. There exists a choice of M as function of L: $K = [\log_2 L] - [\log_2 \log_2 [\log_2 L]]$ where $\log_2$ is the logarithm to the base 2, such that

(i) $$\lim_{L\to\infty} \frac{1}{(\log L)^\gamma} \frac{1}{2^M} \sum_{p=2}^{K} R_p(\mathbb{E}(|J|)) = 0 \tag{2.26}$$

(ii) $$\lim_{L\to\infty} \frac{1}{(\log L)^\gamma} \frac{1}{2^M} \sum_{p=2}^{K} R_p(|J| - \mathbb{E}(|J|)) = 0 \qquad \text{almost surely} \tag{2.27}$$

(iii) $$\text{Prob}\left(\overline{\lim_{L}} \max_{\theta(\Lambda(2L))} \frac{1}{(\log L)^\gamma} \left| \sum_{p=2}^{K} R_p(J, \cos^{(M)} \theta) \right| = 0\right) = 1 \tag{2.28}$$

Part (iii) is the most complicated of this paper and is rather different from the Step 3 of Ref. 1. The main difference is the following: In Ref. 1 we prove some probability estimates which can be used until the width of the crown is 0 (log L). After that we estimate the remainder terms uniformly with respect to J and θ. Here, even if the random variables J are bounded, such a uniform estimate is useful only if this width is 0 (log log L). In this step we improve the probability estimates of Ref. 1 in order that they can be used if the width of crowns is 0 (log log L). This was done by using a different discretization of the cosine and by estimating the distribution of the random variable $\max_\theta(\Delta H(\theta(\mathscr{C}_j^{K+1}), \theta(\mathscr{C}_j^{K+1})))$ without subdividing the crowns $\mathscr{C}_j^{K+1}$ into small squares as in Ref. 1. These two facts are rather technical but are crucial. Moreover this improvement does not give directly the almost sure convergence of $(\log L)^{-\gamma} \max_{\theta(\Lambda(2L))} |\sum_{p=2}^{K} R_p(J, \cos^M \theta)|$ to zero but merely the convergence in probability. We get almost sure convergence when $K(L)$ is such that the width of the smallest crown is $O(\log L)$. To solve this problem we used a method which is very standard in probability theory: First we prove that, for a convenient subsequence L_n, $L_n \to \infty$, with probability 1,

$$\lim_{n\to\infty} \frac{1}{(\log L_n)^\gamma} \max_{\theta(\Lambda(2L_n))} \left| \sum_{p=K_1(L_n)}^{K_2(L_n)} R_p^{(M)} \right| = 0 \tag{2.29}$$

where $K_1(L_n)$ [resp. $K_2(L_n)$] is such that the width of crowns is $O(\log L_n)$ [resp. $O(\log\log L_n)$]. Seconddly we prove that the whole sequence is arbitrary near this subsequence, i.e., with probability 1:

$$\max_{L_m \leqslant L < L_{n+1}} \frac{1}{(\log L_n)^\gamma} \max_{\theta(\Lambda(2L_{n+1}))} \left| \sum_{p=K_1(L_{n+1})}^{K_2(L_{n+1})} R_p - \sum_{p=K_1(L)}^{K_2(L)} R_p \right|$$

is arbitrary small, infinitly often.

This last fact is proven by using an adaptation of the Chung lemma,[13] which is equivalent (see Lemma 2.3) to prove an analog of the Skorokhod maximal inequality.[14]

Step 4. In this step we consider the first two sums in (2.22), calling them $\Delta H_K(J, \cos\theta)$.

Using

$$|\Delta H_K(J, \cos\theta)| \leqslant \Delta H_K(\mathbb{E}(|J|)) + \Delta H_K(|J| - \mathbb{E}(|J|)) \tag{2.30}$$

we prove the following:

Lemma 2.7. If $K_2 = [\log_2 L] - [\log_2 \log_2 [\log_2 L]]$,

(i) $$\lim_{L\to\infty} \frac{1}{(\log L)^\gamma} |\Delta H_{K_2}(|J| - \mathbb{E}(|J|)| = 0 \qquad \text{almost surely} \tag{2.31}$$

(ii) $$\lim_{L\to\infty} \frac{1}{(\log L)^\gamma} \Delta H_{K_2}(\mathbb{E}(|J|) = 0 \qquad \text{for suitable choice of } K(\beta) \tag{2.32}$$

Here also the proof of (2.31) is based on an analog of the Chung lemma.[13] The reason is the following: With our hypothesis on random variables it can be proved that

$$\mathrm{Prob}(|\Delta H_K(|J| - \mathbb{E}(|J|)| \geqslant \delta(\log L)^\gamma) \leqslant \mathrm{ct}\exp[-c(\log L)^\gamma \delta]$$

but since γ is strictly smaller than one we cannot use directly the Borel–Cantelli lemma. On the other hand if J are sub-Gaussian random variables it is easy to see that

$$\mathrm{Prob}(|\Delta H_K(|J| - \mathbb{E}(|J|)| \geqslant \delta(\log L)^\gamma) \leqslant c' \exp[-c\delta^2(\log L)^{2\gamma}]$$

and we can use the Borel Cantelli lemma.

The analog of the Chung lemma allows us to prove Lemma 2.7 in the whole generality of our hypothesis. This is the main difference with Step 4 of Ref. 1 where we proved only analog of 2.32.

Remark. In Step 3, even if the random variable J was sub-Gaussian we need to use the analog of the Chung lemma. Mainly because in Step 3 we prove that it is equivalent to consider random variables which are sub-Gaussian or satisfy our hypothesis. In Step 4 this is not the case.

3. PROOF OF THE PREVIOUS LEMMA

We denote by C a constant which may be different from time to time.

Step 1

Proof of Lemma 2.4. Part (i). Since $\mathbb{E}(J^2) = \sigma^2$ we get $\mathbb{E}(|J|) \leqslant \sigma$. If $B_n = \{x \in \mathbb{Z}^2/|x| = n\}$ it is straightforward that

$$\sum_{x \in B_j, y \in B_{q+j}} \frac{1}{|x-y|^{3+2\varepsilon}} \leqslant cj(q)^{-2-2\varepsilon} \tag{3.1}$$

Using $a(y) = 0$ if $|y| > L$ we get

$$\sum_{\substack{x \in \Lambda(L) \\ y \in \Lambda^c(L^2)}} \frac{\mathbb{E}(|J_{xy}|)}{|x-y|^{3+2\varepsilon}} \{\cosh(a(x)) - 1\}$$
$$c\sigma \sum_{j=1}^{L} j \sum_{n=1}^{\infty} \frac{\{\cosh a(j) - 1\}}{(L^2 + n + L - j)^{2+2\varepsilon}} \tag{3.2}$$

The estimate $\exp a(j) \leqslant (L/j)^{K(\beta)} \exp K(\beta)$ implies that the right-hand side of (3.2) does not exceed $c\sigma L^{-4\varepsilon}$ if $K(\beta) < 2$. This proves (2.12).

Part (ii). Let $\Delta H(\Lambda(L), \Lambda^c(L^2), |J| - \mathbb{E}(|J|))$ be

$$\sum_{\substack{x \in \Lambda(L) \\ y \in \Lambda^c(L^2)}} \frac{(|J_{xy}| - \mathbb{E}(|J_{xy}|))}{|x-y|^{3+2\varepsilon}} \{\cosh(a(x)) - 1\} \tag{3.3}$$

by similar computations we get: if $K(\beta) < 1$

$$\mathbb{E}(\Delta H(\Lambda(L), \Lambda^c(L^2), |J| - \mathbb{E}(|J|))^2 \leqslant c\sigma^2 L^{-6-4\varepsilon}$$

The Tchebychev inequality implies

$$\text{Prob}(|\Delta H(\Lambda(L), \Lambda^c(L^2), |J' - \mathbb{E}(|J||\geqslant \varepsilon_1) \leqslant \frac{c\sigma^2}{\varepsilon_1^2} L^{-6-4\varepsilon} \tag{3.4}$$

Since the series with general term $L^{-6-4\varepsilon}$ is summable the first Borel Cantelli lemma implies 2.13.

Step 2

Proof of Lemma 2.5 part (i) *and* (ii). (i) By similar computations it is straightforward that

$$\Delta H(\Lambda(L), \Lambda(L^2)\backslash\Lambda(2L), \mathbb{E}(|J|) \leqslant \sigma c L^{1-2\varepsilon} \tag{3.5}$$

if $K(\beta) < 2$ and $\varepsilon < 1/2$, if $K(\beta) < 2$ and $\varepsilon = 1/2$ the right-hand side of (3.5) have to be replaced by $\sigma c'$. Therefore if we choose $M = [\log_2 L]$ we get (2.19).

(ii) It is straightforward that if $K(\beta) < 1$

$$\mathbb{E}(\Delta H(\Lambda(L), \Lambda(L^2)\backslash\Lambda(2L), |J| - \mathbb{E}(|J|))^2 \leqslant \sigma^2 c L^{-2-4\varepsilon} \tag{3.6}$$

The Tchebychev inequality and the first Borel Cantelli lemma imply (2.20).

The proof of Lemma 2.5 part (iii) is based on the following exponential estimates for the distribution of sums of independent random variables $(J_K)_{K=1}^n$ which satisfy

$$\mathbb{E}(J_k) = 0, \qquad \mathbb{E}(J_K^2) = \sigma_K^2, \qquad \mathbb{E}(J_K^p) \leqslant p! \frac{\sigma_K^2}{2} H^{p-2}$$

for some fixed $H \in \mathbb{R}^+$ see Ref. 12.

Bernstein Inequality. Let $D_n^2 = \sum_{K=1}^n \sigma_K^2$, then if $S_n = \sum_{K=1}^n J_K$

$$\text{Prob}(|S_n| \geqslant 2xD_n) \leqslant 2\exp(-x^2) \qquad \text{if} \quad 0 < x \leqslant \frac{D_n}{2M} \tag{3.7}$$

$$\text{Prob}(|S_n| \geqslant 2xD_n) \leqslant 2\exp\left(-\frac{x}{2M}\right) \qquad \text{if} \quad x > \frac{D_n}{2M} \tag{3.8}$$

Let us remark that if J_K are sub-Gaussian random variables (3.7) is true for any $X > 0$. With our hypothesis on random variables J we have to be very careful in our summation by blocks if we want to be in a region where (3.7) can be used.

We subdivide $\Lambda(L^2)$ in square of side $2L$. Accoding to this subdivision $\Delta H^{(M)}(\theta(\Lambda(L)), \theta(\Lambda(L^2) \backslash \Lambda(2L)))$ can be written

$$\sum_{\Delta_i \subset \Lambda(L^2) \backslash \Lambda(2L)} \Delta H^{(M)}(\theta(\Lambda(L)), \theta(\Lambda_i)) \tag{3.9}$$

Let us remark that the number of terms in the previous sum does not exceed L^2.

We estimate the distribution of $\Delta H^{(M)}(\theta(\Lambda(L)), \theta(\Lambda_i))$ by using the following lemma which is an adaptation of Lemma 1 of Ref. 10.

Lemma 3.1. Oet Z_i be the center of Λ_i and δ a constant $0 < \delta < 1/2$. If $K(\beta) \leqslant 1 - 2\delta$ then there two constants c, c' such that for any $\mu \in]0, 1[$

$$\text{Prob}\left(\exists \theta(\Lambda(L)), \theta(\Lambda_i)/|\Delta H^{(M)}(\theta(\Lambda(L)), \theta(\Lambda_i))| \geqslant \mu \frac{L^{3+2\delta}}{|Z_i|^{3+2\varepsilon}}\right)$$

$$\leqslant c' \exp -\frac{\mu^2}{8c} L^{2+4\delta} \tag{3.10}$$

Proof of Lemma 3.1. $\Delta H^{(M)}(\theta(\Lambda(L)), \theta(\Lambda_i))$ is equal to

$$\sum_{\substack{x \in \Lambda(L) \\ y \in \Lambda_i}} \frac{J_{xy}}{|x-y|^{3+2\varepsilon}} \left[\sum_{K=0}^{M} \frac{\sigma_K(\theta_x)}{2^{K+1}}\right]\left[\sum_{K'=0}^{M} \frac{\sigma'_K(\theta_y)}{2^{K'+1}}\right] \{\cosh a(x) - 1\} \tag{3.11}$$

If we perform the product $[\sum_{K=0}^{M}][\sum_{K'=0}^{M}]$ we get

$$\Delta H^{(M)} = \sum_{K,K'=0}^{M} \frac{\Delta H_{K,K'}}{2^{K+K'+2}} \tag{3.12}$$

This is the sum of $(M+1)^2$ strongly dependent random variable $\Delta H_{K,K'}$. Given $(K, K') \in \{0,..., M\} \times \{0,..., M\}$ and $\theta(\Lambda(L)), \theta(\Lambda_i)$ a configuration of spins, if $x \in \Lambda(L)$ and $y \in \Lambda_i$ we define the random variable:

$$\eta(x, y) = \frac{J_{xy}}{|x-y|^{3+2\varepsilon}} \sigma_K(\theta_x) \sigma_{K'}(\theta_y) \frac{\{\cosh a(x) - 1\}}{L^{K(\beta)}} |Z_i|^{3+2\varepsilon} \tag{3.13}$$

It is easy to see that $\mathbb{E}(|\eta^p|) \leqslant c^p \mathbb{E}(|J|^p)$; therefore η satisfies the conditions of the Bernstein inequality with some constant $H' = cH$.

Let D^2 be the variance of $\sum_{x \in \Lambda(L), y \in \Lambda_i} \eta(x, y)$. Using $\cosh X - 1 \geqslant X^2/2$, we get

$$
\begin{aligned}
D^2 &\geqslant \frac{cL^2}{2L^{2K(\beta)}} \sigma^2 \sum_{x \in \Lambda(L)} a(x)^4 \\
&\geqslant \frac{c\sigma^2}{2} L^{2-2K(\beta)} \sum_{j=1}^{L/2} j \left[\sum_{K=j}^{L} \frac{1}{L(\log^+ K)^{1-\gamma}} \right]^4 K(\beta)^4 \\
&\geqslant cK^4(\beta)\, \sigma^2 L^{4-2K(\beta)} (\log L)^{-4(1-\gamma)}
\end{aligned} \tag{3.14}
$$

Moreover, it is straightforward that

$$
D^2 \leqslant cL^{4-2K(\beta)} \exp 2K(\beta) \qquad \text{if} \quad K(\beta) < 1 \tag{3.15}
$$

Let us choose X in the Bernstein inequality as

$$
X = \mu \frac{L^{3+2\delta}}{2DL^{K(\beta)}}
$$

Using (3.14) we get $X \leqslant (\mu/2)\, cL^{1+2\delta} (\log L)^{2(1-\gamma)}$ and $D \geqslant c' L^{2-K(\beta)} (\log L)^{-2(1-\gamma)}$. Therefore if $K(\beta) < 1 - 2\delta$ and L is big enough we get $X \leqslant D/4H$ for any fixed H. Therefore we can use (3.7). Using (3.15) we obtain $x \geqslant (\mu/2\sqrt{c})\, L^{1+2\delta}$ and (3.7) leads to the following: for any $(K, K') \in \{0,..., M\} \times \{0,..., M\}$

$$
\text{Prob}\left(|\Delta H_{K,K'}| \geqslant \mu \frac{L^{3+2\delta}}{|Z_i|^{3+2\varepsilon}} \right) \leqslant 2 \exp\left(-\frac{\mu^2}{4c} L^{2+4\delta} \right) \tag{3.16}
$$

Now since the number of random variables we can obtain from $\Delta H_{K,K'}$ by changing $\sigma_K(\theta_x)\, \sigma_{K'}(\theta_y)$ does not exceed 2^{2L^2} we get

$$
\begin{aligned}
&\text{Prob}\left(\exists \theta(\Lambda(L)), \theta(\Lambda_i) / |\Delta H_{K,K'}| \geqslant \frac{\mu L^{3+2\delta}}{|Z_i|^{3+2\varepsilon}} \right) \\
&\qquad \leqslant 2 2^{2L^2} \exp\left(-\frac{\mu^2}{4c} L^{2+4\delta} \right) \leqslant c' \exp\left(-\frac{\mu^2}{8c} L^{2+4\delta} \right)
\end{aligned} \tag{3.17}
$$

If we use the fact that

$$
\begin{aligned}
&\Big\{ \exists \theta(\Lambda(L)), \theta(\Lambda_i) / |\Delta H^{(M)}(\theta(\Lambda(L)), \theta(\Lambda_i))| \\
&\qquad \geqslant \frac{\mu L^{3+2\delta}}{|Z_i|^{3+2\varepsilon}} \left(\sum_{K,K'=0}^{M} \frac{1}{2^{K+K'+2}} \right) \Big\}
\end{aligned} \tag{3.18}
$$

is contained in

$$\bigcup_{K,K'=0}^{M} \left\{ \exists\theta(\Lambda(L)), \theta(\Lambda_i)/|\Delta H_{K,K'}| \geqslant \mu \frac{L^{3+2\delta}}{|Z_i|^{3+2\varepsilon}} \frac{1}{2^{K+K'+2}} \right\} \tag{3.19}$$

(as it can be easily checked by taking the complement) we get

$$\text{Prob}\left(\exists\theta(\Lambda(L)), \theta(\Lambda_i)/|\Delta H^{(M)}(\theta(\Lambda(L)), \theta(\Lambda_i)| \geqslant 4\mu \frac{L^{3+2\delta}}{|Z_i|^{3+2\varepsilon}} \right)$$
$$\leqslant (M+1)^2 \exp\left(-\frac{\mu^2}{8c} L^{2+4\delta} \right) \leqslant c' \exp\left(-\frac{\mu^2}{16c} L^{2+4\delta} \right) \tag{3.20}$$

the last inequality in (3.20) comes from $M = 0$ ($\log L$).

Remark. This is at this point that the discretization of $\cos\theta_x$ instead of θ_x is crucial: we need a lower bound on the covariance if we want to use (3.7). Moreover we obtain a better estimate than in Ref. 1.

We can pove Lemma 2.5, part (iii).

Using similar statements as (3.23) $\subset$ (3.24) or Lemma 3.3 of Ref. 1 we get:

$$\text{Prob}\Bigg(\exists\theta(\Lambda(L)), \theta(\Lambda(L^2)\backslash\Lambda(2L))/|\Delta H^{(M)}(\theta(\Lambda(L)), \theta(\Lambda(L^2)\backslash\Lambda(2L))|$$
$$\geqslant 4\mu L^{3+2\delta} \sum_i \frac{1}{|Z_i|^{3+2\varepsilon}} \Bigg)$$
$$\leqslant c' 2L^2 \exp\left(-\frac{\mu^2}{16c} L^{2+4\delta} \right) \leqslant c'' \exp\left(-\frac{\mu^2}{32c} L^{2+4\delta} \right) \tag{3.21}$$

because there is no more than L^2 boxes Λ_i in $\Lambda(L^2)\backslash\Lambda(2L)$. Now by a simple scaling

$$\sum_i \frac{1}{|Z_i|^{3+2\varepsilon}} \leqslant \frac{c}{(L)^{3+2\varepsilon}} \tag{3.22}$$

Therefore if we choose $0 < \delta < \varepsilon$ and L large enough (in order that $L^{\delta-\varepsilon} c \leqslant 1$) we get

$$\text{Prob}(\exists\theta(\Lambda(L)), \theta(\Lambda(L^2)\backslash\Lambda(2L)/\Delta H^{(M)} \geqslant \mu)$$
$$\leqslant c' \exp -\frac{\mu^2}{32c} L^{2+4\delta} \tag{3.23}$$

which together with the first Borel Cantelli lemma implies (2.21).

Step 3

We prove the result only for the first sum in (2.23). The second sum is treated in the same way.

Proof of Lemma 2.6. (i) Letting $p \in 1, 2,..., K$ and $j \in 1, 2,..., \alpha_p(L)$ we consider

$$\Delta H(\mathscr{C}_j^{p+1}, \mathscr{C}_{j+2}^{p+1}, \mathbb{E}(|J|)) = \sum_{\substack{x \in \mathscr{C}_j^{p+1} \\ y \in \mathscr{C}_{j+2}^{p+1}}} \frac{\mathbb{E}(|J_{xy}|)}{|x-y|^{3+2\varepsilon}} \{\cosh(a(x)-a(y))-1\} \tag{3.24}$$

Assume $2 \leqslant j \leqslant \alpha_p(L) - 3$. Using (3.1) we get

$$\Delta H(\mathscr{C}_j^{p+1}, \mathscr{C}_{j+2}^{p+1}, \mathbb{E}(|J|)) \leqslant c\sigma \sum_{q=(j-1)2^{m-p}+1}^{j2^{m-p}} q \sum_{K=(j+1)2^{m-p}+1}^{(j+2)2^{m-p}} \frac{\{\cosh[a(q)-a(K)]-1\}}{(K-q)^{2+2\varepsilon}} \tag{3.25}$$

On the other hand,

$$|a(q)-a(K)| \leqslant K(\beta) \frac{(K-q)}{q(\log q)^{1-\gamma}} \tag{3.26}$$

Since in (3.25) $q \geqslant (j-1)2^{m-p}+1$ and $K \leqslant (j+2)2^{m-p}$ the right-hand side of (3.26) does not exceed $[1+3/(j-1)]\,K(\beta)$. Therefore for some constant c_1

$$\{\cosh[a(q)-a(K)]-1\} \leqslant c_1 K^2(\beta) \frac{(K-q)^2}{q^2(\log q)^{2(1-\gamma)}} \tag{3.27}$$

Inserting (3.27) in (3.25) we get

$$(3.25) \leqslant c\sigma^2 K^2(\beta)(2^{m-p})^{1-2\varepsilon} \sum_{q=(j-1)2^{m-p}+1}^{j2^{m-p}} \frac{1}{q(\log q)^{2(1-\gamma)}} \tag{3.28}$$

If $\alpha_p(L)-2 \leqslant j \leqslant \alpha_p(L)-1$ it is straight forward that the same result is true up to a multiplicative constant, for $j=\alpha_p(L)$ the sum in (3.28) runs from $(\alpha_p(L)-1)2^{m-p}+1$ to L. If $j=1$ we remark that

$$|a(q)-a(K)| \leqslant K(\beta) \left\{\log \frac{(K+2^{m-p+1})}{q} + \frac{1}{q(\log^+ q)^{1-\gamma}}\right\} \tag{3.29}$$

which follows by comparison with an integral. Using (3.29) it is straightforward that

$$\Delta H(\mathscr{C}_1^{p+1}, \mathscr{C}_3^{p+2}, \mathbb{E}(|J|)) \leqslant \sigma c(2^{m-p})^{1-2\varepsilon} \exp K(\beta) \tag{3.30}$$

(3.30) together with (3.28) leads to

$$\sum_{j=1}^{\alpha_p(L)} \Delta H(\mathscr{C}_j^{p+2}, \mathscr{C}_{j+2}^{p+2}, \mathbb{E}(|J|)) \leqslant \sigma c(2^{m-p})^{1-2\varepsilon} \sum_{j=1}^{L} \frac{1}{q(\log^+ q)^{2(1-\gamma)}}$$

Hence

$$\sum_{p=2}^{K} R_p(\mathbb{E}(|J|)) \leqslant \sigma c(2^m)^{1-2\varepsilon}(\log L)^{2\gamma-1} \tag{3.31}$$

If we choose $M = [\log_2 L]$ we get (2.26).

We prove Lemma 2.6, part (ii):

By similar computations we get

$$\mathbb{E}(\Delta H(\mathscr{C}_j^{p+1}, \mathscr{C}_{j+2}^{p+1}, |J| - \mathbb{E}(|J|))^2$$

$$\leqslant c\sigma^2(2^{m-p})^{-4\varepsilon} \sum_{q=(j-1)2^{m-p}+1}^{j2^{m-p}} \frac{1}{q^3(\log^+ q)^{4(1-\gamma)}} \quad \text{if } 2 \leqslant j \leqslant \alpha_p(L) - 1 \tag{3.32}$$

$$c'\sigma^2(2^{m-p})^{-4\varepsilon} \sum_{q=(\alpha_p(L)-1)2^{m-p}+1}^{L} \frac{1}{q^3(\log^+ q)^{4(1-\gamma)}} \quad \text{if } j = \alpha_p(L) \tag{3.33}$$

$$c''\sigma^2(2^{m-p})^{-2-4\varepsilon} \qquad \text{if } \quad j = 1 \tag{3.34}$$

Therefore

$$\mathbb{E}(R_p^2(|J| - \mathbb{E}(|J|)) \leqslant \sigma^2 c(2^{m-p})^{-4\varepsilon} \tag{3.35}$$

The Tchebychev inequality leads to

$$\text{Prob}\left(\frac{1}{(\log L)^\gamma} \frac{1}{2^M} \left(\sum_{p=2}^{K} R_p(|J| - \mathbb{E}(|J|)) \geqslant \varepsilon_1\right)\right.$$

$$\leqslant \frac{\sigma^2 c}{\varepsilon_1^2(\log L)^{2\gamma}} \frac{1}{2^{2M}} \sum_{p=2}^{K} \frac{1}{(2^{m-p})^{4\varepsilon}} \tag{3.36}$$

Since $2^{2M} \geqslant c'L^2$ $\sum_{p=2}^{K} (2^{m-p})^{-4\varepsilon} \leqslant c''K(2^{K-m})^{4\varepsilon}$ and by construction $K \leqslant m = O(\log L)$, the first Borel Cantelli lemma implies (2.27). In order to prove Lemma 2.6, part (iii) we need the following lemma, which is an improvement of Lemma II.5 of Ref. 1.

Lemma 3.2. For any $\delta > 0$, $\varepsilon_1 \in]0, 1]$ there exist constants c, c', such that if $K(\beta)/\gamma < 1$ then

$$\begin{aligned}\operatorname{Prob}(\exists\theta(\mathscr{C}_j^{p+1}), \theta(\mathscr{C}_{j+2}^{p+1})/|\Delta H^{(M)}(\theta(\mathscr{C}_j^{p+1}), \theta(\mathscr{C}_{j+2}^{p+1})| \\ \geqslant \varepsilon_1(2^{-m+p})^{-2(\varepsilon-\delta)} f_j(2^{m-p})) \\ \leqslant c'(\log 2^m)^2 \exp\left(-\frac{\varepsilon_1^2}{c}\right)(2^{m-p})^{2+2\delta} \max(1, j-1)\end{aligned} \tag{3.37}$$

where

$$f_j(2^{m-p}) = [\max(1, j-1)(\log^+(1+(j-1)2^{m-p})^{2(1-\gamma)}]^{-1}$$

Proof of Lemma 3.2. We assume $j \leqslant a_p(L) - 3$.

As in the beginning of the proof of Lemma 3.1 for any given configuration of spins $\theta(\mathscr{C}_j^{p+1})$, $\theta(\mathscr{C}_{j+2}^{p+1})$, $\Delta H^{(M)}(\theta(\mathscr{C}_j^{p+2}), \theta(\mathscr{C}_{j+2}^{p+1}))$ can be written

$$\sum_{K,K'=0}^{M} \frac{\Delta H_{K,K'}}{2^{K+K'+2}}(\mathscr{C}_j^{p+2}, \mathscr{C}_{j+2}^{p+1})$$

For any $\{K, K'\} \in \{0,..., M\} \times \{0,..., M\}$ if $\theta(\mathscr{C}_j^{p+1})$, $\theta(\mathscr{C}_{j+2}^{p+1})$ is a configuration of spin and $x \in \mathscr{C}_j^{p+1}$, $y \in \mathscr{C}_{j+2}^{p+1}$ are lattice sites we define the following random variable:

$$\eta(x, y) = \frac{J_{xy}}{|x-y|^{3+2\varepsilon}} \sigma_K(\theta_x)\, \sigma_{K'}(\theta_y)\{\cosh[a(x) - a(y)] - 1\}\, g_j(2^{m-p}) \tag{3.38}$$

where

$$g_j(2^{m-p}) = \begin{cases} (2^{m-p})^{1+2\varepsilon-K(\beta)/\gamma} & \text{if } j = 1 \\ (j-1)^{3/2}(2^{m-p})^{1+2\varepsilon}\{\log[1+(j-1)2^{m-p}]\}^{2(1-\gamma)} & \text{if } j \geqslant 2 \end{cases} \tag{3.39}$$

Using if $j \geqslant 2$,

$$|a(x) - a(y)| \leqslant K(\beta)(j-1)^{-1}[\log 1 + (j-1)2^{m-p}]^{\gamma-1} \tag{3.40}$$

if $j = 1$,

$$|a(x) - a(y)| \leqslant K(\beta) + \frac{K(\beta)}{\gamma} \log 3.2^{m-p} \tag{3.41}$$

it is easy to check that

$$|\eta(x, y)| \leqslant \frac{c\,|J_{xy}|}{(\max(1, j-1))^{1/2}} (2^{m-p})^2 \tag{3.42}$$

this estimate together with the hypothesis on random variable J leads to

$$\mathbb{E}(\exp t\eta(x, y)) \leqslant 1 + \sigma^2 \frac{t^2}{2} g_j^2(2^{m-p}) \left[\frac{\cosh(a(x)-a(y))-1}{|x-y|^{3+2\varepsilon}}\right]^2 \times \left[1 + \sum_{l=3}^{\infty} t^{l-2} H^{l-2} \left(\frac{c}{\max(1, j-1)^{1/2}(2^{m-p})^2}\right)^{l-2}\right] \tag{3.43}$$

If

$$t \leqslant \frac{(\max(1, j-1)^{1/2}(2^{m-p})^2)}{2Hc} \tag{3.44}$$

the series into the last brackets does not exceed 2. Therefore if (3.44) occurs

$$\mathbb{E}(\exp t\eta(x, y)) \leqslant \exp t^2\sigma^2 g_j^2(2^{m-p}) \left\{\frac{\cosh(a(x)-a(y))-1}{|x-y|^{3+2\varepsilon}}\right\}^2 \tag{3.45}$$

Using the fact that for different values of x, y $\eta(x, y)$ are independent and by some straightforward calculations we get the following:

If (3.44) occurs,

$$\mathbb{E}\left(\exp t \sum_{\substack{x \in \mathscr{C}_j^{p+1} \\ y \in \mathscr{C}_{j+2}^{p+1}}} \eta(x, y)\right) \leqslant \exp t^2\sigma^2 c_j \tag{3.46}$$

where, for some constants c, c',

$$c_j = c \qquad \text{if } \ j \geqslant 2$$

$$c_1 \leqslant c'(2^{m-p})^{-2K(\beta)/\gamma} \qquad \text{if } \ j = 1$$

If we set $t = \tau/g_j(2^{m-p})$ and we used the fact that

$$\Delta H_{K,K'} = \frac{1}{g_j(2^{m-p})} \sum_{x,y} \eta(x, y)$$

We get

$$\mathbb{E}(\exp \tau \Delta H_{K,K'}) \leqslant \exp \frac{\tau^2\sigma^2 c_j}{g_j^2(2^{m-p})} \tag{3.47}$$

if

$$\tau \leqslant \frac{g_j(2^{m-p})}{2Mc}(\max(1, j-1))^{1/2}(2^{m-p})^2 \tag{3.48}$$

The Markov inequality leads to

$$\begin{aligned} &\mathrm{Prob}(\Delta H_{K,K'} \geqslant \varepsilon_1 f_j(2^{m-p})(2^{m-p})^{-2(\varepsilon-\delta)}) \\ &\quad \leqslant \exp\left\{-\tau\varepsilon_1 \frac{f_j(2^{m-p})}{(2^{m-p})^{2(\varepsilon-\delta)}} + \frac{\tau^2 c_j \sigma^2}{g_j^2(2^{m-p})}\right\} \end{aligned} \tag{3.49}$$

We choose

$$\tau = \frac{\varepsilon_1}{2Hc} g_j(2^{m-p})(\max(1, j-1))^{1/2} h_j(2^{m-p})$$

where

$$h_j(2^{m-p}) = \begin{cases} 2^{m-p} & \text{if } j \geqslant 2 \\ (2^{m-p})^{1+K(\beta)/\gamma} & \text{if } j = 1 \end{cases}$$

It is straightforward that (3.48) occurs if $K(\beta)/\gamma \leqslant 1$. On the other hand some easy computations show that the right-hand side of (3.49) does not exceed $c' \exp[-\varepsilon_1^2(\max(1, j-1))(2^{m-p})^{2+2\delta}]/4Hc$ if 2^{m-p} is large enough.

Since if we change $\sigma_K(\theta_x)\,\sigma_{K'}(\theta_y)$ we can obtain no more than $\exp(8_j + 8)(2^{m-p})^2 \log 2$ different random variables $\Delta H_{K,K'}$, we get

$$\begin{aligned} &\mathrm{Prob}(\exists\theta(\mathscr{C}_j^{p+1}), \theta(\mathscr{C}_{j+2}^{p+1})/\Delta H_{K,K'} \geqslant \varepsilon_1 f_j(2^{m-p})(2^{m-p})^{-2(\varepsilon-\delta)}) \\ &\quad \leqslant c' \exp\left\{(8_j + 8)(2^{m-p})^2 \log 2 - \frac{\varepsilon_1^2}{4Hc} \max(1, j-1)(2^{m-p})^{2+2\delta}\right\} \end{aligned} \tag{3.50}$$

If m is large enough, since $p \leqslant m - [\log_2 \log_2 m]$, $m - p$ goes to infinity with m, the right-hand side of (3.50) does not exceed

$$c'' \exp -\frac{\varepsilon_1^2}{8Hc}(\max(1, j-1))(2^{m-p})^{2+2\delta}$$

Using now the fact that

$$\Delta H^{(M)}(\theta(\mathscr{C}_j^{p+1}), \theta(\mathscr{C}_{j+2}^{p+1})) = \sum_{K,K'=0}^{M} \frac{\Delta H_{K,K'}}{2^{K+K'+2}}$$

$M = [\log_2 L] = m(L)$ and similar argument as (3.23) $\subset$ (3.24), we get II.38.

If $\alpha_p(L)-2 \leqslant j \leqslant \alpha_p(L)$ each step of the previous proofs can be checked with some trivial change of constants c, c' and ε_1 become $\varepsilon_1 \tilde{c}$ for some constant $\tilde{c}$ independent of p.

The proof of Lemma 2.6, part, (iii) was done in two parts: in the first part we consider the term

$$\frac{1}{(\log L)^{\gamma}} \sum_{p=2}^{K_1(L)} R_p \qquad \text{where } K_1(L) = [\log_2 L] - [\log_2[\log_2 L]] \tag{3.51}$$

In the second part we consider separately

$$\bar{R}_L = \frac{1}{(\log L)^{\gamma}} \sum_{p=K_1(K)+1}^{K_2(L)} \left(\sum_{j=\alpha_{p-1}(L)-2}^{\alpha_{p-1}(L)} \Delta H^{(M)}(\theta(\mathscr{C}_j^p), \theta(\mathscr{C}_{j+2}^p)) \tag{3.52}$$

and

$$\frac{1}{(\log L)^{\gamma}} \sum_{p=K_1(L)+1}^{K_2(L)} R_p - \bar{R}_L \tag{3.53}$$

where $K_2(L) = [\log_2 L] - [\log_2 \log_2[\log_2 L]]$. Let us remark that $K_1(L)$ and $K_2(L)$ are constant if $2^m \leqslant L \leqslant 2^{m+1}-1$.

First Part. With the help of an argument similar to (3.28) ⊂ (3.19) (or Lemma II.3 of Ref. 1) and the two following estimates:

(1) $$\sum_{J=1}^{\alpha_{p-1}(L)} [\max(1, j-1)]^{-1} [\log^+(1+(j-1)2^{m-p})]^{-2(1-\gamma)}$$
$$\leqslant c(\log L)^{-1+2\gamma} \log\log L \tag{3.54}$$

(2) $$\sum_{j=1}^{\alpha_{p-1}(L)} \exp\left[-\frac{\varepsilon_1^2}{c}(2^{m-p})^{2+2\delta} \max(1, j-1)\right]$$
$$\leqslant c' \exp\left[-\frac{\varepsilon_1^2}{c}(2^{m-p})^{2+2\delta}\right] \tag{3.55}$$

Lemma 3.2 leads to

$$\text{Prob}\left(\exists\theta(\Lambda(2L)) \middle/ \frac{|R_p|}{(\log L)^{\gamma}} \geqslant \varepsilon_1 \frac{\log\log L}{(\log L)^{1-\gamma}(2^{m-p})^{2(\varepsilon-\delta)}}\right)$$
$$c(\log 2^m)^2 \exp\left[-\frac{\varepsilon_1^2}{c}(2^{m-p})^{2+2\delta}\right] \tag{3.56}$$

Using one more time a similar argument to (3.18) ⊂ (3.19) and the two following estimates:

$$\text{(i)} \quad \sum_{p=2}^{K_1(L)} (2^{m-p})^{-2(\varepsilon-\delta)} \leqslant c(2^{m-K_1(L)})^{-2(\varepsilon-\delta)} \tag{3.57}$$

$$\text{(ii)} \quad \sum_{p=2}^{K_1(L)} \exp\left[-\frac{\varepsilon_1^2}{2}(2^{m-p})^{2+2\delta}\right] \leqslant c \log L \exp\left\{-\frac{\varepsilon_1^2}{c}([\log_2 L])^{2+2\delta}\right\} \tag{3.58}$$

inequality (3.56) leads to

$$\begin{aligned}\text{Prob}\left(\exists\theta(\Lambda(2L))\Big/\sum_{p=2}^{K_1(L)} \frac{R_p}{(\log L)^\gamma} \geqslant \varepsilon_1 \frac{(\log \log L)}{(\log L)^{1-\gamma+2(\varepsilon-\delta)}}\right.\\ \leqslant c'(\log L)^3 \exp\left\{-\frac{\varepsilon_1^2}{c}([\log_2 L])^{2+2\delta}\right\}\end{aligned} \tag{3.59}$$

If m is large enough

$$\frac{\log \log L}{(\log L)^{1-\gamma+2(\varepsilon-\delta)}} \leqslant 1$$

The first Borel Cantelli lemma and (3.59) leads to

$$\lim_{L\to\infty} \frac{1}{(\log L)^\gamma} \max_{\theta(\Lambda(2L))} \left|\sum_{p=2}^{K_1(L)} R_p\right| = 0 \qquad \text{almost surely}$$

We prove the second part.

Let us first remark that we cannot use directly estimates (3.56) because it can be checked that

$$\begin{aligned}\text{Prob}\left(\exists\theta(\Lambda(2L))\Big/\sum_{p=K_1(L)}^{K_2(L)} R_p \geqslant \varepsilon_1(\log L)^\gamma\right)\\ \leqslant c(\log L)^3 \exp\left\{-(2\log_2[\log_2 L])^{2+2\delta}\frac{\varepsilon_1^2}{c}\right\}\end{aligned} \tag{3.60}$$

and the series with general term the right-hand side of (3.60) is not summable. On the other hand (3.60) implies that

$$\lim_{L\to\infty} \frac{1}{(\log L)^\gamma} \max_{\theta(\Lambda(2L))} \sum_{p=K_1(L)}^{K_2(L)} R_p = 0 \qquad \text{in probablity} \tag{3.61}$$

Moreover of we consider the subsequence $L_m = 2^m$ (3.60) together with the first Borel Cantelli lemma leads to

$$\lim_{m\to\infty} \frac{1}{(\log 2^m)^\gamma} \max_{\theta(\Lambda(2^m))} \sum_{p=K_1(2^m)}^{K_2(2^m)} R_p = 0 \tag{3.62}$$

almost surely. Since the width of the crown $\mathscr{C}^p_{\alpha_p(L)}$ changes when L varies between 2^m and 2^{m+1} we consider the terms (3.52) which contain all the terms where $\mathscr{C}^p_{\alpha_p(L)}$ occurs.

Using (3.37), the fact that $K_2(L)-K_1(L)\leqslant c(\log L)$ and $(\alpha_p(L)-2)2^{m-p}\geqslant cL$ for some constant $0<c<1$ if $p\in K_1(L)$, $K_2(L)$ it is straightforward that

$$\begin{aligned}&\text{Prob}(\exists\theta(\Lambda(2L))/\bar{R}_L\geqslant\varepsilon_1)\\&\quad\leqslant c'(\log L)^3\exp\left[-\frac{\varepsilon_1^2}{2}(L)(2^{m-p})^{1+2\delta}\right]\\&\quad\leqslant c''\exp\left(-\frac{\varepsilon_1^2}{2}L\right)\end{aligned}\tag{3.63}$$

if L is large enough, which together with the first Borel Cantelli lemma leads to

$$\lim_{L\to\infty}\max_{\theta(\Lambda_{2L})}\bar{R}_L=0\qquad\text{almost surely}$$

We consider (3.53). Let $\tilde{R}_p=R_p-\sum_{j=\alpha_{p-1}(L)-2}^{\alpha_{p-1}(L)}\Delta H(\theta(\mathscr{C}^p_j),\ \theta(\mathscr{C}^p_{j+2}))$. We prove an adaptation of the Chung theorem[13] which would imply that (3.61) and (3.62) lead to

$$\lim_{L\to\infty}\frac{1}{(\log L)^\gamma}\max_{\theta(\Lambda(2L))}\sum_{p=K_1(L)}^{K_2(L)}\tilde{R}_p=0\qquad\text{almost surely}\tag{3.64}$$

Let us explain the strategy of the proof: Let $K_1(L)$ and $K_2(L)$ be as before. Let $K, K'\in\{0,\ldots,M\}\times\{0,\ldots,M\}$ and $\theta(\Lambda(2L))$ a given configuration of spins. We define the following:

$$\text{(i)}\quad A_L(\sigma_K,\sigma_{K'})=\left\{\frac{1}{(\log L)^\gamma}\sum_{p=K_1}^{K_2}\tilde{R}_p(\sigma_K\sigma_{K'})\geqslant\varepsilon_1\right\}\tag{3.65}$$

where $\tilde{R}_p(\sigma_K,\sigma_K)$ corresponds to a given configuration of $\sigma_K(\theta_x)\,\sigma_{K'}(\theta_y)$;

$$\text{(ii)}\quad A_L(K,K')=\bigcup_{\{\sigma_K,\sigma_{K'}\}}A_L(\sigma_K,\sigma_{K'})\tag{3.66}$$

where the union runs over all the possible configurations of $\sigma_K,\sigma_{K'}$;

$$\text{(iii)}\quad A_L=\sum_{K,K'=0}^{M}A_L(K,K')\tag{3.67}$$

We want to prove

$$\lim_{L_0 \to \infty} \text{Prob}\left(\bigcup_{L \geqslant L_0} A_L \right) = 0 \tag{3.68}$$

We can assume $L_0 = 2^{p_0}$. Since $\bigcup_{L \geqslant L_0} A_L$ can be written $\bigcup_{m=p_0}^{\infty} \{ \bigcup_{L=2^m}^{2^{m+1}-1} A_L \}$, we get

$$\text{Prob}\left(\bigcup_{L \geqslant L_0} A_L \right) \leqslant \sum_{m=p_0}^{\infty} \text{Prob}\left(\bigcup_{L=2^m}^{2^{m+1}-1} A_L \right) \tag{3.69}$$

Therefore if we can prove that $\text{Prob}(\bigcup_{L=2^m}^{2^{m+1}-1} A_L)$ is the general term of a summable series we get (3.68).

Let us first remark that $M(L) = [\log_2 L]$ is constant and equal to m if $L \in I_m = [2^m, ..., 2^{m+1} - 1]$. Therefore it is straightforward that

$$\text{Prob}\left(\bigcup_{L=2^m}^{2^{m+1}-1} \left\{ \bigcup_{K,K'=0}^{M(L)} A_L(K, K') \right\} \right) \leqslant \sum_{K,K'=0}^{M} \text{Prob}\left(\bigcup_{L=2^m}^{2^{m+1}-1} A_L(K, K') \right) \tag{3.70}$$

On the other hand if $2^m \leqslant L < L' \leqslant 2^{m+1} - 1$ then $\tilde{R}_p\{\sigma_K, \sigma_{K'}\}(L') - \tilde{R}_p\{\sigma_K, \sigma_{K'}\}(L)$ and $\tilde{R}_p\{\sigma_K, \sigma_{K'}\}(L)$, $\tilde{R}_p\{\sigma_K, \sigma_{K'}\}(L-1) \cdots \tilde{R}_p\{\sigma_K \sigma_{K'}\}(2^m)$ are independent random variables. This follows from the fact that $\tilde{R}_p\{\sigma_K, \sigma_{K'}\}(L)$ and $\tilde{R}_p\{\sigma_K, \sigma_{K'}\}(L')$ are sums of independent random variables indexed by a family of crowns of width 2^{m-p-1} and since $L' > L$ the summation runs over more terms in $\tilde{R}_p\{\sigma_K, \sigma_{K'}\}(L')$ then in $\tilde{R}_p\{\sigma_K, \sigma_{K'}\}(L)$. The following lemma is an analog of the Skorokhod maximal inequality.[14]

Lemma 3.3. Let $0 < \varepsilon_1 \leqslant 1$ be an arbitrary real number. Let j be an integer $1 \leqslant j \leqslant 2^m - 1$.

If

$$B_j(\{\sigma_K, \sigma_{K'}\}) = \left\{ \left| \sum_{p=K_1}^{K_2} \tilde{R}_p\{\sigma_K, \sigma_{K'}\}(2^m + j) - \sum_{p=K_1}^{K_2} \tilde{R}_p\{\sigma_K, \sigma_{K'}\}(2^{m+1} - 1) \right| \leqslant \varepsilon_1 (\log 2^m)^\gamma \right\}$$

$$C_j(\{\sigma_K, \sigma_{K'}\}) = \left\{ \left| \sum_{p=K_1}^{K_2} \tilde{R}_p\{\sigma_K, \sigma_{K'}\}(2^m + j) \right| \geqslant 2\varepsilon_1 (\log 2^m)^\gamma \right\}$$

$$\hat{B}_j = \bigcap_{\{\sigma_K, \sigma_{K'}\}} B_j(\{\sigma_K, \sigma_{K'}\}), \qquad \hat{C}_j = \bigcup_{\{\sigma_K, \sigma_{K'}\}} C_j(\{\sigma_K, \sigma_{K'}\})$$

then

$$\text{Prob}\left(\bigcup_{\{\sigma_K,\sigma_{K'}\}} A_{2^{m+1}-1}\{\sigma_K,\sigma_{K'}\}\right) \geqslant \left(\inf_{1\leqslant j\leqslant 2^m-1} \text{Prob}(\hat{B}_j)\right) \text{Prob}\left(\bigcup_{j=1}^{2^m-1} \hat{C}_j\right) \tag{3.71}$$

Proof of Lemma 3.2. By the triangle inequality we get, for any $j \in \{1, 2,..., 2^m - 1\}$,

$$A_{2^{m+1}-1}(\{\sigma_K,\sigma_{K'}\}) \supset B_j(\{\sigma_K,\sigma_{K'}\}) \cap C_j(\{\sigma_K,\sigma_{K'}\}) \tag{3.72}$$

Therefore

$$A_{2^{m+1}-1}(K, K') \supset \bigcup_{j=1}^{2^m-1} (\hat{B}_j \cap \hat{C}_j) \tag{3.73}$$

Since we have seen that $B_j(\{\sigma_K,\sigma_{K'}\})$ and $C_j\{\sigma_K,\sigma_{K'}) C_{j-1}\{\sigma_x,\sigma_{K'}\} \cdots C_1\{\sigma_K,\sigma_{K'}\}$ are independent events, it is straightforward that $\hat{B}_j$ and $\hat{C}_j$, $\hat{C}_{j-1},..., \hat{C}_1$ are independent events. Therefore if we set $C_0 = \phi$, $B_0 = \phi$, we get the following chain of inequalities

$$\begin{aligned} \text{Prob}(A_{2^{m+1}-1}(K,K')) &\geqslant \text{Prob}\left(\bigcup_{j=1}^{2^m-1} \hat{B}_j \cap \hat{C}_j\right) \qquad [\text{by } (3.74)] \\ &= \sum_{j=1}^{2^m-1} \text{Prob}\left((\hat{B}_j \cap \hat{C}_j) \bigcap_{n=0}^{j-1} (\hat{B}_n \cap \hat{C}_n)^c\right) \\ &\geqslant \sum_{j=1}^{2^m-1} \text{Prob}\left((\hat{B}_j \cap \hat{C}_j) \bigcap_{n=0}^{j-1} (\hat{C}_n)^c\right) \\ &= \sum_{j=1}^{2^m-1} \text{Prob}(\hat{B}_j)\, \text{Prob}\left(\hat{C}_j \bigcap_{n=0}^{j-1} (\hat{C}_n)^c\right) \qquad (\text{by independence}) \\ &\geqslant \left(\inf_{1\leqslant j\leqslant 2^m-1} \text{Prob}(\hat{B}_j)\right) \text{Prob}\left(\bigcup_{j=1}^{2^m-1} \hat{C}_j\right) \end{aligned} \tag{3.74}$$

and this proves Lemma 3.3.

Now we can prove 3.64: Since $\text{Prob}(\hat{B}_j^c)$ does not exceed

$$\begin{aligned} &\text{Prob}\left(\exists\{\sigma_K,\sigma_{K'}\} \Big/ \left| \sum_{p=K_2}^{K_2} \tilde{R}_p\{\sigma_K,\sigma_{K'}\}(2^m + j) \geqslant \frac{\varepsilon_1}{2} (\log 2^m)^\gamma\right.\right) \\ &\quad + \text{Prob}\left(\exists\{\sigma_K,\sigma_{K'}\} \Big/ \left| \sum_{p=K_1}^{K_2} \tilde{R}_p\{\sigma_K,\sigma_{K'}\}(2^{m+1} - 1) \geqslant \frac{\varepsilon_1}{2} (\log 2^m)^\gamma\right.\right) \end{aligned} \tag{3.75}$$

(3.60) leads to the following: for any $j \in 1,\ldots, 2^m - 1$,

$$\operatorname{Prob}(\hat{B}_j) \geqslant 1 - c \exp -\frac{\varepsilon_1^2}{2}(2[\log_2 \log_2 2^m])^{2+2\delta} \tag{3.76}$$

if m is large enough.

Therefore Lemma 3.3 and (3.60), (3.70), (3.76) lead to if m is large enough:

$$\operatorname{Prob}\left(\bigcup_{L=2^m}^{2^{m+1}-1} A_L\right) \leqslant c'' \exp -\frac{\varepsilon_1^2}{c}(2[\log_2 \log_2 2^m])^{2+2\delta} \tag{3.77}$$

Since the right hand side of (3.78) is the general term of summable series, we get the result.

Step 4

Proof of Lemma 2.7. Part (ii). Using (3.1) and the analog to (3.26), it is not difficult to check that

$$\frac{1}{(\log L)^\gamma} \sum_{j=2}^{\alpha_{K_2}(L)} \Delta H(\mathscr{C}_j^{K_2+1}, \mathscr{C}_j^{K_2+1} \cup \mathscr{C}_{j+1}^{K_2+1}, \mathbb{E}(|J|)) \tag{3.78}$$

does not exceed $c(\log\log L)^{2\varepsilon-1}(\log L)^{\gamma-1}$ which goes to zero when L goes to infinity if $1-\gamma > 0$.

On the other hand,

$$\frac{1}{(\log L)^\gamma} \Delta H(\mathscr{C}_1^{K_2+1}, \mathscr{C}_1^{K_2+1} \cup \mathscr{C}_2^{K_2+1}, \mathbb{E}(|J|)$$

$$\leqslant c\sigma(\log L)^{-\gamma} \sum_{s=1}^{2^{m-K_2}} s \sum_{t=s+1}^{2^{m-K_2+1}} \frac{1}{(t-s)^{2+2\varepsilon}} \{\cosh(a(t)-a(s))-1\} \tag{3.79}$$

The sum $\sum_{t=s+1}^{2^{m-K_2+1}}$ can be written $\sum_{t=s+1}^{2s} + \sum_{t=2s+1}^{2^{m-K_2+1}}$; if $s+1 \leqslant t \leqslant 2s$ it is straightforward that

$$|a(t)-a(s)| \leqslant 2K(\beta)(\log s)^{\gamma-1} \tag{3.80}$$

and therefore $\{\cosh(a(t)-a(s))-1\} \leqslant c(a(t)-a(s))^2$. If $t \geqslant 2s+1$ we used

$$|a(t)-a(s)| \leqslant K(\beta)\left(\frac{1}{s} + \log\frac{t}{s}\right)$$

which follows by comparison with an integral. These two facts and some computation lead to (3.79) $\leqslant c\sigma(2^{m-K_2})^{1-2\varepsilon+K(\beta)}(\log L)^{\gamma}$, which goes to zero if L goes to infinity because $2^{m-K_2}=O(\log\log L)$.

Part (i). If we use similar summation by blocks as in previous proof it is not difficult to check that

$$\mathbb{E}(\Delta H_{K_2}^2(|J|-\mathbb{E}(|J|))\leqslant\sigma^2 c' \tag{3.81}$$

for some constant c'.

The Tchebytchev inequality yields

$$\text{Prob}(\Delta H_{K_2}(|J|-\mathbb{E}(|J|))\geqslant\varepsilon_1(\log L)^{\gamma})\leqslant\frac{c'\sigma 2}{\varepsilon_1^2(\log L)^{2\gamma}} \tag{3.82}$$

We use here an adaptation of the Chung lemma. As in Step 3 we first consider the term where $\mathscr{C}^{K_2+1}_{\alpha_{K_2}(L)}$ occurs:

$$\overline{\Delta H}_{K_2}=\sum_{j=\alpha_{K_2}-2}^{\alpha_{K_2}}\Delta H_{K_2}(\mathscr{C}_j^{k_2+1},\mathscr{C}_j^{K_2+1}\cup\mathscr{C}_{j+1}^{K_1+1},|J|-\mathbb{E}(|J|))$$

it is straightforward that

$$\mathbb{E}(\overline{\Delta H}_{K_2})^2\leqslant c(2^{m-K_2})/[(\alpha_{K_2(L)-2})2^{m-K_2}]^3 \tag{3.83}$$

Using $\alpha_{K_2(L)}2^{m-K_2}\geqslant c'L$ for some constant $0<c'<1$ if m is big enough. The Tchebychev inequality and the first Borel Cantelli lemma leads to

$$\lim_{L\to\infty}\frac{1}{(\log L)^{\gamma}}\overline{\Delta H}_K=0\qquad\text{almost surely}$$

Now we have to prove that

$$\lim_{L\to\infty}\frac{1}{(\log L)^{\gamma}}(\Delta H_{K_2}-\overline{\Delta H}_K)=0\qquad\text{almost surely}$$

Let $\widetilde{\Delta H}_{K_2}=\Delta H_{K_2}-\overline{\Delta H}_K$. If $2^m\leqslant L<L'\leqslant 2^{m+1}-1$ then $\widetilde{\Delta H}_{K_2}(L)-\widetilde{\Delta H}_{K_2}(L')$ is independent of $\widetilde{\Delta H}_{K_2}(L)$, $\widetilde{\Delta H}_{K_2}(L-1)\cdots\widetilde{\Delta H}_{K_2}(2^m)$ because here also $\widetilde{\Delta H}_{K_2}(L)$ and $\widetilde{\Delta H}_{K_2}(L')$ are sums of independent random variable indexed by crowns of width 2^{m-K_2} and since $L'>L$ the summation runs over more terms in $\widetilde{\Delta H}_{K_2}(L')$ than in $\widetilde{\Delta H}_{K_2}(L)$. At this point the proof is exactly the same as the end of the Step 3 with the following modification in Lemma 3.3:

$$\hat{B}_j = \{|\widetilde{\Delta H}_{K_2}(2^m + j) - \widetilde{\Delta H}_{K_2}(2^{m+1} - 1)| \leqslant \varepsilon_1(\log 2^m)^\gamma\} \tag{3.84}$$

$$\hat{C}_j = \{|\widetilde{\Delta H}_{K_2}(2^m + j)| \geqslant 2\varepsilon_1(\log 2^m)^\gamma\} \tag{3.85}$$

$$A_{2^{m+1}-1} = \{|\widetilde{\Delta H}_{K_2}(2^{m+1} - 1)| \geqslant \varepsilon_1(\log 2^m)^\gamma\} \tag{3.86}$$

and writing

$$\text{Prob}(A_{2^{m+1}-1}) \geqslant \text{Prob}\left(\bigcup_{j=1}^{2^m-1} \hat{B}_j \cap \hat{C}_j\right] \tag{3.87}$$

instead of (3.74), we get

$$\text{Prob}(A_{2^{m+1}-1}) \geqslant \left[\operatorname*{Inf}_{1 \leqslant j \leqslant 2^m - 1} \text{Prob}(\hat{B}_j)\right) \text{Prob}\left(\bigcup_{j=1}^{2^m-1} \hat{C}_j\right) \tag{3.88}$$

Using (3.81) and the Tchebyshev inequality it is straightforward that

$$\text{Prob}(\hat{B}_j^c) \leqslant c(\log 2^m)^{-2\gamma} \tag{3.89}$$

Therefore (3.88) and (3.89) lead to

$$\text{Prob}\left(\bigcup_{j=1}^{2^m-1} \hat{C}_j\right) \leqslant \frac{c}{(m \log 2)^{2\gamma}} [1 - c(m \log 2)^{-2\gamma}]^{-1} \tag{3.90}$$

Since $2\gamma > 1$, the right-hand side of (3.90) is the general term of a summable serie and from the first Borel Cantelli lemma we get the result.

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REFERENCES

1. P. Picco, On the absence of breakdown of symmetry for the plane rotator model with long range random interaction, preprint, Marseille, November 1982), to appear in *J. Stat. Phys.*
2. D. Jasnow and M. E. Fisher, *Phys. Rev. B* **3**:895, 907 (1971).
3. O. A. McBryan and T. Spencer, *Commun. Math. Phys.* **53**:299 (1977).
4. S. B. Shlosman, *Teor. Mat. Fiz.* **37**:1118 (1978).
5. C. A. Bonato, J. F. Perez, and A. Klein, *J. Stat. Phys.* **29**:159 (1982).

6. K. R. Ito, *J. Stat. Phys.* **29**:747 (1982).
7. A. Messager, S. Miracle-Sole, and J. Ruiz, Upper bound on the decay of correlation in $SO(N)$ symmetric spin systems with long range interaction, preprint, Marseille, November 1982.
8. J. Frohlich, R. Israel, E. H. Lieb, and B. Simon, *Commun. Math. Phys.* **62**:1 (1978).
9. C. E. Pfister, *Commun. Math. Phys.* **79**:181 (1981).
10. K. M. Khanin and Ya. G. Sinai, *J. Stat. Phys.* **20**:573 (1979).
11. K. M. Khanin, *Teor. Mat. Fiz.* **43**:253 (1980).
12. V. V. Petrov, *Sums of Independent Random Variables* (Springer, New York, 1975), p. 55.
13. Y. S. Chow and H. Teicher, *Probability Theory* (Springer, New York, 1978).
14. L. Breiman, *Probability* (Addison-Wesley, Reading, Massachusetts, 1968).